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LETTER TO THE EDITOR

On some exactly solvable potentials derived from supersymmetric quantum mechanics

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Abstract. It is demonstrated that seven exactly solvable potentials derived recently from supersymmetric quantum mechanics and presented as new ones can, in fact, be obtained from one or other of the 12 known shape-invariant potentials by a simple choice of the potential parameters. The remarkable result that the energy spectrum of three potentials is independent of certain parameters is also interpreted in a straightforward way.

The introduction of supersymmetric quantum mechanics (SUSYQM) [1] generated renewed interest in solvable problems of non-relativistic quantum mechanics. This approach relates pairs of one-dimensional potentials ('SUSY partner potentials') to each other using (super)algebraic manipulations, and has led to the remarkable finding that knowing the ground-state wavefunction of a potential $V_-(x)$, one can easily construct its SUSY partner $V_+(x)$ with the same energy eigenvalues, except for the ground state, which is missing from the spectrum of $V_+(x)$. (See, for example [2] for a recent review.) Several new exactly solvable potentials were identified this way.

It was soon noticed that the SUSY partner potentials often depend on the coordinate in the same way [3], which simplifies the calculation of the wavefunctions and energy eigenvalues to a considerable extent. These 'shape-invariant' potentials, which form a special subclass of solvable potentials were, in fact, found to be essentially the same as those obtained earlier [4] from the factorization method. Several attempts were made to identify and classify all the shape-invariant potentials [5–7], and the results suggest that finding further such potentials in addition to the 12 known ones is unlikely, nevertheless its possibility cannot be ruled out completely.

Recently a number of new solvable potentials have been reported either as basic new results or as examples or illustrations to some novel approaches inspired by SUSYQM. Here we show that although the authors stress the novelty [8–11] and the shape-invariant nature [10, 11] of these potentials, they can be obtained from already known PI and PII class [6] (or, alternatively [4], type A and E) shape-invariant potentials by an appropriate choice of the potential parameters. A common feature of these potentials is that their solutions contain Jacobi polynomials (which are closely related to the hypergeometric functions [12]). We summarized the relation between the 'new' and 'old' potentials in table 2. Using the notation of [6] the latter can be written formally as $V_-(A, B; ax + \delta)$, where A and B define the shape of the potentials, a is a scaling factor of the coordinate, while δ simply corresponds to a shift along the x axis. We displayed the 'new' potentials and the parameters they depend on in table 1, and gave their relation to the 'old' potentials $V_{-}(A, B; ax + \delta)$ in table 2. We also presented the notation of the potentials in terms of a classification scheme of shape-invariant potentials described in [6]. Here $g(x)$ is a function which appears in the argument of the Jacobi polynomials and which identifies uniquely the individual potentials within the PI and PII classes [6]. (This classification scheme is basically the same as the one obtained from the factorization method [4], although the two approaches follow different principles of classification, and it differs from a third scheme [5] which is based on the formalism of SUSYQM.)

Table 1. The 'new' potentials in [8, 9, 10, 11] expressed in their original form. See the text for the definition of the 'deformed hyperbolic functions' $\mathrm{sech}_q\, x$ and $\tanh_q x$ in [10].

Reference	$V(x)$	parameters
[8]	$-(\rho^2\cos 2\phi - \frac{1}{4})\,\mathrm{sech}^2 x + \rho^2 \sin 2\phi\, \mathrm{sech}^2 x \sinh x$	ρ, ϕ
[9]	$-\frac{1}{4}(1-\alpha^2)\frac{a^2\exp(-2x)}{(2-a\exp(-x))^2} - \frac{Aa\exp(-x)}{2-a\exp(-x)}$	α, A, a
[10] Ex. 1.	$-\lambda(\lambda-a)q(\mathrm{sech}_q\,\alpha x)^2 + 2\beta\tanh_q \alpha x + \lambda^2 + \frac{\beta^2}{\lambda^2}$	α, β, λ, q
Ex. 2.	$(\mu^2 - q\lambda(\lambda-\alpha))(\mathrm{sech}_q\,\alpha x)^2 + \mu(2\lambda-\alpha)(\tanh_q \alpha x)(\mathrm{sech}_q\,\alpha x) + \lambda^2$	α, λ, μ, q
[11] (33)	$(a+b)^2 - \frac{a(a+1)}{\cosh^2 x} + \frac{b(b+1)}{\sinh^2 x}$	a, b
(39)	$-a^2 + \frac{b^2}{a^2} + \frac{a(a-1)}{\cos^2 x} + 2b\tan x$	a, b
(40)	$-a^2 + \frac{b^2+a(a-1)}{\cos^2 x} + (2a-1)b\frac{\sin x}{\cos^2 x}$	a, b

Table 2. The relation between the 'new' potentials and the corresponding 'old' shape-invariant potentials. (Notation of [6] were used here.) One should avoid confusing parameters A and a in [9] and [11] with A and a in [6].

$V_{-}(A, B; u)$, $u = ax + \delta$	A	B	a	δ	Class $(g(x))$
$A^2 + \frac{B^2 - A(A+a)}{\cosh^2 u} + \frac{B(2A+a)\sinh u}{\cosh^2 u}$	$\rho\cos\phi - \frac{1}{2}$	$\rho\sin\phi$	1	0	$\mathrm{PI}(\mathrm{i}\sinh x)$
$A^2 + \frac{B^2}{A^2} + \frac{A(A-a)}{\sinh^2 u} - 2B\coth u$	$-\frac{\alpha+1}{4}$	$\frac{1-\alpha^2}{16} - \frac{A}{4}$	$-\frac{1}{2}$	$\frac{1}{2}\ln\frac{a}{2}$	$\mathrm{PII}(\coth x)$
$A^2 + \frac{B^2}{A^2} - \frac{A(A+a)}{\cosh^2 u} + 2B\tanh u$	$-\lambda$	β	α	$-\frac{1}{2}\ln q$	$\mathrm{PII}(\tanh x)$
$A^2 + \frac{B^2 - A(A+a)}{\cosh^2 u} + \frac{B(2A+a)\sinh u}{\cosh^2 u}$	$-\lambda$	$-\mu q^{-1/2}$	α	$-\frac{1}{2}\ln q$	$\mathrm{PI}(\mathrm{i}\sinh x)$
$(A-B)^2 - \frac{A(A+a)}{\cosh^2 u} + \frac{B(B-a)}{\sinh^2 u}$	a	$-b$	1	0	$\mathrm{PI}(\cosh 2x)$
$-A^2 + \frac{B^2}{A^2} + \frac{A(A+a)}{\sin^2 u} - 2B\cot u$	$-a$	b	1	$\frac{\pi}{2}$	$\mathrm{PII}(-\mathrm{i}\cot x)$
$-A^2 + \frac{B^2 + A(A-a)}{\sin^2 u} - \frac{B(2A-a)\cos u}{\sin^2 u}$	a	$-b$	1	$\frac{\pi}{2}$	$\mathrm{PI}(\cos x)$

In order to get the conventions usually used in SUSYQM (i.e. to get a potential with zero ground-state energy) one also has to add a constant term to the potentials in [8] and [9]. In addition to this, one also has to rewrite hyperbolic functions in terms of exponential ones to recover the potential discussed by Williams [9] in its original form. The expression of the energy eigenvalues E_n also has to be rearranged somewhat in this case.

The 'new' potentials were obtained using various approaches. Williams [9] followed earlier works [6, 13] to construct a new class of solvable potentials related to the Jacobi polynomials but, as can be seen from the tables, what he found is a PII class (or type E) potential, sometimes referred to as the Eckart potential, shifted along the x axis. In a recent study of the mathematical background of SUSYQM and shape-invariance Arai [10] used 'deformed hyperbolic functions' (like $\sinh_q x \equiv \frac{1}{2}(e^x - q\, e^{-x})$, $\cosh_q x \equiv \frac{1}{2}(e^x + q\, e^{-x})$, $\mathrm{sech}_q\, x \equiv 1/\cosh_q x$ and $\tanh_q x \equiv \sinh_q x/\cosh_q x$) to derive 'new' shape-invariant potentials as illustrative examples. It can be shown, however, that these are also shifted PI and PII class potentials. Both Williams [9] and Arai [10] notice that the energy spectrum of their potentials is independent of the a and q parameters, respectively. The interpretation of this result is clear since, as we see, these parameters represent practically only a shift along the x axis, so they should not be relevant to the basic results. (It has to be mentioned, though, that B depends on q in Example 2 of [10] but, similarly to other PI class [6] (or type A [4]) potentials the energy spectrum is independent of B, and therefore of q and μ.)

Cerveró [8] used a new method (based on the Jacobi elliptic functions) to derive solvable potentials within the framework of SUSYQM, and taking the hyperbolic functions as a special case (the standard limit of the Jacobi elliptic functions,) he obtained a two-parameter potential which contains the Pöschl–Teller potential as a special limit. Noting that this latter one is known to possess zero reflection coefficient in certain cases, Cerveró also suggested that the new potential maintains this property for some values of the parameters since, as he claimed, it can be obtained from the Pöschl–Teller potential through supersymmetric transformations. However, as can be seen from the tables, this is a PI class potential and was known earlier. Furthermore, scattering amplitudes have also been determined for this case [14], and the results show that it can be reflectionless only if A/a is an integer and $B = 0$ hold, i.e. only in case of the symmetric Pöschl–Teller potential with specific depth. It can also be shown that one cannot obtain the general potential in [8] from the symmetric, reflectionless Pöschl–Teller potential by means of supersymmetric transformations.

Cao [11] presented a number of two-parameter shape-invariant potentials to demonstrate that they can be recovered from the new '$\hat{C}$ transformation'. There are three 'new' ones among them which, however, can be obtained from 'old' shape-invariant potentials by elementary transformations. Although potentials (39) and (40) in [11] formally differ from the ones in earlier compilations [4–6, 15], sine and cosine functions can be rewritten into each other by simply adding $\pi/2$ to their arguments (i.e. by a shift of the coordinate x), which means that they cannot be considered essentially new potentials. We mention here that potential (39) in [11] has also been referred to briefly as a new one by Barclay and Maxwell [7].

The potentials discussed here all belong to the PI and PII class, which implies that the corresponding wavefunctions can be expressed in terms of the $P_n^{(\alpha,\beta)}(z)$ Jacobi polynomials. The two parameters give rise to a wide variety of potential shapes, which may explain why some of the 'new' potentials failed to be identified with 'old' ones. A remarkable fact is that three of the five PI class, and all three PII class potentials were derived in one way or another without being related to earlier results.

We remark here that some PII class potentials are in close connection with the Coulomb problem. Noting, for example, that the $\cosh x$ function is close to 1 near the origin, while the $\sinh x$ function approximates x there, the Eckart potential (PII($\coth x$)) with $x > 0$ approximates $V(x) = -Ze^2/x + l(l+1)/x^2$ (up to an additive constant) if we write $2B/a = Ze^2$ and $A/a = l+1$ (see table 1.). This is a reasonable approximation only if the wavefunctions are restricted to the neighbourhood of the origin, but this is the case for large values of B and relatively small values of A and n. This result is also confirmed by the energy spectrum $E_n^{(-)} = A^2 + B^2/A^2 - (A+na)^2 - B^2/(A+na)^2$ which is dominated by the last term in this case, recovering the familiar $-Z^2e^4/4(n+l+1)^2$ expression. (Similar considerations are valid for the PII $(-\mathrm{i}\cot x)$ potential in table 2 too.) An interesting aspect of this connection is that the Hulthén potential is sometimes compared with the Coulomb problem [16]. Their relation becomes transparent in view of the fact that the Eckart (PII $(\coth x)$) potential contains the $V(r) = -V_0 \exp(-r/c)/(1-\exp(-r/c))$ Hulthén potential as a special case, with $A = a = \frac{1}{2}c$ and $B = \frac{1}{4}V_0$, (and, of course, $x = r$). This connection between the PII class [6] (or type E [4]) potentials (like the Eckart and Rosen–Morse potentials) and the Coulomb problem, which is an LIII [6] (or type F [4]) potential is also confirmed by the classification scheme of shape-invariant potentials presented by Cooper *et al* [5], according to which the above three potentials belong to the same class. The PII $(-\mathrm{i}\cot x)$ potential, which is present in [4], but is missing from [5] (as well as from other compilations [15],) could also be assigned to this common class.

In conclusion, we have shown that some shape-invariant [10, 11], and non-shape-invariant but solvable potentials [8, 9] presented recently as new potentials can, in fact, be obtained from one or another of the 12 known shape-invariant potentials. This result is in accordance with the conclusion drawn from several independent studies [5–7] using essentially different methods, implying that finding further shape-invariant potentials is unlikely, although a complete proof of this conjecture has not been given yet. Despite this negative result, the variety of ways the 'new' potentials were found shows the potential power of SUSYQM to generate further exactly solvable (but possibly not shape-invariant) potentials.

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